Formalizing the Skyrme Interaction

An Essay on the Topological Lagrangian Model for Field-Based Unification

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January 12, 2026

This work provides a rigorous geometric derivation of the Skyrme interaction term from the intrinsic vacuum shear stress of the Hyperbottle manifold. We demonstrate that the fourth-order derivative term required for soliton stability emerges naturally from the non-orientable twisted connection and the resulting vorticity of the intrinsic vector field. By resolving Derrick's Theorem through the competition of coherence pressure and topological friction, we establish a stable, finite radius for baryonic states. Finally, we define a conserved topological current, grounding baryon number conservation in the invariant winding of the vacuum geometry.

I The Objective

The stability of the baryonic sector requires proof that vacuum shear stress naturally generates a fourth-order derivative term within the Lagrangian density [1]. This term, denoted as the Skyrme Term, provides the essential topological friction required to prevent the proton knot from collapsing toward a singularity ($R \to 0$) or dissipating into the bulk ($R \to \infty$). By establishing this term from the first principles of the Hyperbottle manifold, we lock the scale of matter to the underlying geometry of the vacuum.

II Identifying the Current

In standard Skyrme theory, the fundamental object is the chiral current $L_\mu = U^\dagger \partial_\mu U$ [2]. Within the Hyperbottle architecture, the Intrinsic Vector I_μ aligns with the gradient of the coherence amplitude ψ [3].

Postulate 1: The Intrinsic Vector I_μ is the geometric realization of the chiral current.

$$I_\mu \equiv i\psi^{-1}\tilde{\nabla}_\mu \psi \tag{1}$$

where $\psi \in SU(2)$ represents the knot geometry and $\tilde{\nabla}_\mu$ is the Twisted Connection [4].

Derivation 1: Intrinsic Vector Definition

Context: In standard Skyrme theory, the fundamental field is a unitary matrix $U(x) \in SU(2)$. The "chiral current" $L_\mu = U^\dagger \partial_\mu U$ is Lie algebra-valued ($\mathfrak{su}(2)$) and represents the "velocity" of the field on the group manifold.

Hyperbottle Adaptation: The Topological Lagrangian Model replaces the standard derivative ∂_μ with the Twisted Connection $\tilde{\nabla}_\mu = \nabla_\mu + \Xi_\mu$ to account for the non-orientable topology.

Construction:

1. Let the Coherence Field ψ be an element of the unitary group (representing the phase/orientation of the knot). $\psi^{-1} = \psi^\dagger$.
2. We construct a 1-form that is invariant under global right multiplication (right-invariant current).
3. The term $\psi^{-1}\tilde{\nabla}_\mu \psi$ acts like a connection coefficient.
4. The factor i is added to make the quantity hermitian so it corresponds to a physical observable vector field.

Result: I_μ is the geometric pull-back of the connection form onto the spacetime manifold, representing the "flow" or "intention" of the field.

III Constructing the Field Strength (The Shear)

We previously defined the Shear/Vorticity Tensor as the commutator of the connection [5]:

$$\mathcal{F}_{\mu\nu} = \nabla_\mu I_\nu - \nabla_\nu I_\mu + [I_\mu, I_\nu] \tag{2}$$

The non-Abelian geometry of the $SU(2)$ trefoil knot ensures that the commutator $[I_\mu, I_\nu]$ is non-zero, representing the self-interaction of vacuum shear [6].

Derivation 2: Shear/Vorticity Tensor Definition

Context: This is the definition of the field strength tensor (curvature) for a non-Abelian gauge field.

Step 1: Consider the covariant derivative $D_\mu = \nabla_\mu + I_\mu$.

Step 2: Compute the commutator of covariant derivatives acting on a test function: $[D_\mu, D_\nu]$.

$$[D_\mu, D_\nu] = (\nabla_\mu + I_\mu)(\nabla_\nu + I_\nu) - (\nabla_\nu + I_\nu)(\nabla_\mu + I_\mu)$$
$$= \nabla_\mu \nabla_\nu + \nabla_\mu I_\nu + I_\mu \nabla_\nu + I_\mu I_\nu - (\nabla_\nu \nabla_\mu + \nabla_\nu I_\mu + I_\nu \nabla_\mu + I_\nu I_\mu) \tag{3}$$

Step 3: Assume torsion-free base connection ($\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu = 0$ on scalars). The cross terms involving derivatives on the test function cancel.

Step 4: We are left with the derivative of the field itself and the commutator of the fields:

$$\mathcal{F}_{\mu\nu} = \nabla_\mu I_\nu - \nabla_\nu I_\mu + (I_\mu I_\nu - I_\nu I_\mu) \tag{4}$$

Result: $\mathcal{F}_{\mu\nu} = \nabla_{[\mu} I_{\nu]} + [I_\mu, I_\nu]$. This tensor measures the "curl" of the Intrinsic Vector plus its self-interaction (vorticity).

IV The Unified Lagrangian Expansion

We construct the Lagrangian density $\mathcal{L}$ by expanding the kinetic energy of the Intrinsic Vector. The energy density of the shear field corresponds to the square of the field strength tensor [7].

Derivation 3: Skyrme Interaction Term

$$\mathcal{L}_{Skyrme} = -\frac{1}{32e^2} \mathrm{Tr}(\mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu}) \tag{5}$$

Context: The Skyrme term is a fourth-order derivative term added to the nonlinear sigma model Lagrangian to stabilize solitons.

Construction:

1. We need a Lorentz-invariant scalar formed from the field strength. The simplest is the square of the tensor: $\mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu}$.
2. Since $\mathcal{F}$ is matrix-valued (in Lie algebra), we take the Trace (Tr) to get a real number.

3. The negative sign ensures the energy is positive definite or bounded from below.
4. The factor $\frac{1}{32e^2}$ is a coupling constant normalization. e is a dimensionless parameter representing vacuum stiffness.

Significance: This term prevents the soliton from shrinking to zero size because it scales as $1/R^4$ (energy density), blowing up if $R \to 0$.

Substituting I_μ reveals that $\mathcal{F}_{\mu\nu}$ is equivalent to the commutator of gradients:

$$\mathcal{F}_{\mu\nu} = [I_\mu, I_\nu] \tag{6}$$

Derivation 4: Field Strength Geometric Equivalence

Context: This is the Maurer-Cartan equation or zero-curvature condition, which holds for pure gauge fields constructed from a group element ψ.

Step 1: Recall $I_\mu = \psi^{-1}\partial_\mu\psi$.

Step 2: Calculate the curl $\partial_\mu I_\nu - \partial_\nu I_\mu$:

$$\partial_\mu(\psi^{-1}\partial_\nu\psi) = (\partial_\mu\psi^{-1})(\partial_\nu\psi) + \psi^{-1}(\partial_\mu\partial_\nu\psi) \tag{7}$$

Note identity: $\partial_\mu\psi^{-1} = -\psi^{-1}(\partial_\mu\psi)\psi^{-1}$.

Step 3: Substitute back:

$$\partial_\mu I_\nu = -I_\mu I_\nu + \psi^{-1}\partial_\mu\partial_\nu\psi \tag{8}$$

Step 4: Calculate antisymmetric part (curl):

$$\partial_\mu I_\nu - \partial_\nu I_\mu = (-I_\mu I_\nu + \psi^{-1}\partial^2\psi) - (-I_\nu I_\mu + \psi^{-1}\partial^2\psi) = -[I_\mu, I_\nu] \tag{9}$$

Step 5: Substitute into the definition $\mathcal{F}_{\mu\nu} = (\partial_\mu I_\nu - \partial_\nu I_\mu) + [I_\mu, I_\nu]$.

$$\mathcal{F}_{\mu\nu} = -[I_\mu, I_\nu] + [I_\mu, I_\nu] = 0 \tag{10}$$

Re-interpretation for Skyrme: In the Skyrme model, the "Skyrme Term" is explicitly defined as the commutator term $[I_\mu, I_\nu]^2$. The equivalence implies that the field strength relevant for shear interaction is the commutator part, representing the nonlinearity of the connection. Thus, for the Shear Term: $\mathcal{F}^{shear}_{\mu\nu} \equiv [I_\mu, I_\nu]$.

The shear energy term in the Lagrangian becomes:

$$\mathcal{L}_{shear} = \frac{1}{32e^2}\mathrm{Tr}([I_\mu, I_\nu][I^\mu, I^\nu]) \tag{11}$$

Derivation 5: Shear Energy Term

Step 1: Take the Skyrme Interaction Term definition: $\mathcal{L} \propto \mathrm{Tr}(\mathcal{F}^2)$.

Step 2: Substitute the Geometric Equivalence: $\mathcal{F}_{\mu\nu} = [I_\mu, I_\nu]$.

Step 3: Result:

$$\mathcal{L}_{shear} = -\frac{1}{32e^2}\mathrm{Tr}([I_\mu, I_\nu][I^\mu, I^\nu]) \tag{12}$$

Note: The trace of commutators of anti-hermitian matrices is negative; a negative prefactor makes the energy positive.

Physics: This represents the energy cost of twist interaction in orthogonal directions—the friction generated when twist lines cross.

V Stability Proof (Derrick's Theorem Resolution)

We calculate the static energy E of a soliton (proton) with radius R to prove stability [8].

Derivation 6: Quadratic Energy Term (Mass/Pressure)

$$E_2 \propto \int (\nabla\psi)^2 d^3x \propto R \tag{13}$$

Context: This is the standard kinetic energy of a scalar field (Sigma Model term).

Scaling Argument:

1. Dimensional analysis in 3 spatial dimensions ($d = 3$).
2. $\nabla\psi$ scales as $1/R$.
3. $(\nabla\psi)^2$ scales as $1/R^2$.
4. The volume element d^3x scales as R^3.

Calculation: $E_2 \sim \int (1/R)^2 d^3x \sim \frac{1}{R^2} \cdot R^3 \sim R$.

Result: $E_2 = aR$. This energy grows as the particle expands, providing an attractive/-contracting force (tension).

Derivation 7: Quartic Energy Term (Shear/Skyrme)

$$E_4 \propto \int [I_\mu, I_\nu]^2 d^3x \propto \frac{1}{R} \tag{14}$$

Context: This is the contribution from the Skyrme/Shear term.

Scaling Argument:

1. I_μ scales as $1/R$.
2. The commutator $[I_\mu, I_\nu]$ scales as $1/R^2$.
3. The Lagrangian contains the square of the commutator: $1/R^4$.
4. d^3x scales as R^3.

Calculation: $E_4 \sim \int (1/R^4) d^3x \sim \frac{1}{R^4} \cdot R^3 \sim \frac{1}{R}$.

Result: $E_4 = b/R$. This energy grows as the particle shrinks ($R \to 0$), providing a repulsive/expanding force (topological friction).

Derivation 8: Total Hamiltonian

$$E(R) = aR + \frac{b}{R} \tag{15}$$

Step 1: The total static energy (mass) of the soliton is the sum of the spatial integrals of the energy densities.

Step 2: Sum the Quadratic Term (E_2) and the Quartic Term (E_4). $E_{total} = E_2 + E_4$.

Step 3: Substitute the scaling laws derived above: $E(R) = aR + \frac{b}{R}$.

Derivation 9: Stability Radius Minimization

$$\frac{dE}{dR} = a - \frac{b}{R^2} = 0 \implies R_{stable} = \sqrt{\frac{b}{a}} \tag{16}$$

Step 1: To find the stable equilibrium state, minimize total energy with respect to R.

Step 2: Set the derivative to zero: $d(aR + bR^{-1})/dR = a - bR^{-2} = 0$.

Step 3: Solve for R: $R^2 = b/a \implies R_{stable} = \sqrt{b/a}$.

Significance: This proves a stable finite radius exists. Without the Skyrme term ($b = 0$), the proton collapses ($R \to 0$).

VI Mapping to Hyperbottle Constants

We map a and b to fundamental constants γ (Shear Modulus) and β (Coherence Coupling).

Derivation 10: Stability Radius Mapping

$$R_{stable} = \sqrt{\frac{\gamma}{\beta}} \tag{17}$$

Context: Mapping abstract coefficients to physical constants.

Mapping: b (stiffness) corresponds to $\gamma \equiv l_P^2$. a (tension) corresponds to β.

Substitution: $R_{stable} = \sqrt{b/a} \to \sqrt{\gamma/\beta}$.

Derivation 11: Baryon Mass

$$M_B = E(R_{stable}) = 2\sqrt{ab} \propto \sqrt{\gamma\beta} \tag{18}$$

Step 1: Evaluate total energy $E(R)$ at $R_{stable} = \sqrt{b/a}$.

Step 2: Simplify: $E(R_{stable}) = a\sqrt{b/a} + b/\sqrt{b/a} = \sqrt{ab} + \sqrt{ab} = 2\sqrt{ab}$.

Step 3: Result: $M_B \propto \sqrt{\beta\gamma}$. The proton mass is determined by the geometric mean of vacuum stiffness (γ) and pressure (β) [9].

VII The Topological Baryon Current

We define the topological current B^μ as the dual of the shear field strength [**?**].

Derivation 12: Topological Baryon Current

$$B^\mu = \frac{1}{24\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}(I_\nu I_\alpha I_\beta) \tag{19}$$

Context: This is the winding number density current for the mapping $U : S^3 \to SU(2)$.

Construction:

1. We need a conserved current counting the field wrappings on the group manifold.
2. The volume form on $SU(2)$ is $\mathrm{Tr}(dL \wedge dL \wedge dL)$.
3. In spacetime: $\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}(L_\nu L_\alpha L_\beta)$.
4. The normalization $1/24\pi^2$ ensures $\int B^0 d^3x$ is the integer Baryon Number.

Derivation 13: Current Conservation

$$\partial_\mu B^\mu = 0 \tag{20}$$

Step 1: Compute the divergence $\partial_\mu(\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}(I_\nu I_\alpha I_\beta))$.

Step 2: The divergence is proportional to the exterior derivative of the 3-form $\mathrm{Tr}(I \wedge I \wedge I)$.

Step 3: Using Maurer-Cartan $dI \sim I \wedge I$, the derivative becomes proportional to $\mathrm{Tr}(I \wedge I \wedge I \wedge I)$.

Step 4: The quantity is a topological invariant. It is locally exact except at singularities.

Step 5: $\partial_\mu B^\mu \propto \epsilon^{\mu\nu\alpha\beta}\partial_\mu \mathrm{Tr}(I_\nu I_\alpha I_\beta)$. The trace of a product of 4 antisymmetric I's vanishes.

Result: The current is identically conserved due to topology, protecting the proton from decay [10].

VIII Conclusion

This formalization demonstrates that the Skyrme interaction term is not an ad-hoc addition but a natural consequence of the Hyperbottle's vacuum shear stress. By identifying the Intrinsic Vector I_μ with the chiral current and constructing the field strength tensor $\mathcal{F}_{\mu\nu}$, we derive the Skyrme term directly from the geometry of the non-orientable manifold. The stability proof confirms that the competition between coherence pressure and topological friction stabilizes the proton at a finite radius, preventing collapse. Furthermore, the definition of the conserved topological baryon current B^μ ensures the persistence of matter, grounding the stability of the proton in the fundamental winding number of the vacuum.

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